

Are Kaluza-Klein modes enhanced by parametric resonance?

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We study parametric amplification of Kaluza-Klein (KK) modes in a higher D -dimensional generalized Kaluza-Klein theory, which was originally considered by Mukohyama in the narrow resonance case. It was suggested that KK modes can be enhanced by an oscillation of a scale of compactification by the d -dimensional sphere S^d ($d = D - 4$) and by the direct product $S^{d_1} \times S^{d_2}$ ($d_1 + d_2 = D - 4$). We extend this past work to the more general case where initial values of the scale of compactification and the quantum number of the angular momentum l of KK modes are not small. We perform analytic approaches based on the Mathieu equation as well as numerical calculations, and find that the expansion of the universe rapidly makes the KK field deviate from instability bands. As a result, KK modes are not enhanced sufficiently in an expanding universe in these two classes of models.

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I. INTRODUCTION

Recently, much interest has been focused on higher-dimensional theories. The original idea goes back to Kaluza and Klein [1], who considered that the unification of the interactions in Nature may be realized in the gravitational theory in five dimensions. At present, there are many higher dimensional theories which are the generalizations of the Kaluza-Klein theory, for example, $D = 10$ superstring theory [2], and $D = 11$ M-theory [3]. Very recently, the universe of brane models [4] has received much attention as a solution to the hierarchy problem between the weak and Planck scale. In order to hide extra dimensions, we generally assume that they are within a very small compact internal space. Since the typical scale of the internal space is the Planck length $l_{\text{Pl}} \sim 10^{-33}\text{cm}$, which is much smaller than the size of the external space, we can reduce higher-dimensional theories to the familiar four-dimensional gravitational theories.

There are various ways for Kaluza-Klein reductions to four dimensions. As was pointed out in Ref. [5], it is possible to judge whether such reductions are appropriate from a cosmological point of view. For example, in the model where the geometry is described by a direct product of the four-dimensional Minkowski spacetime M^4 and 1-sphere S^1 considered by Kolb and Slansky [6], particles of Kaluza-Klein (KK) modes which are so called *pyrgon* are produced. Since the energy density of this massive particle decreases as $\rho_{KK} \sim a^{-3}$, this density will dominate over the energy density of the universe in the radiation dominant era if pyrgons are overproduced in the early stage of the universe

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and do not decay by entropy productions. Mukohyama [5] extended this model, and considered the compactifications both by a d -dimensional sphere S^d with $d = D - 4$ and by a direct product $S^{d_1} \times S^{d_2}$ with $d_1 + d_2 = D - 4$. In these classes of models, since the reduced effective potential in the four-dimensional theory lacks a global minimum, we need to introduce the Casimir effect as a one-loop quantum correction to give a stable ground state. Then a scalar field σ which corresponds to the scale of the compactification oscillates around the minimum of its potential, and is finally trapped at the ground state $\sigma = 0$ [7]. Since KK modes acquire masses which are expressed by the oscillating σ field, we can expect the amplification of KK modes by parametric resonance. This picture is similar to the scenario of preheating after inflation where scalar fields coupled to an inflaton field are resonantly enhanced in the oscillating stage of inflaton [8,9]. In the case of a chaotic inflation model with a quadratic potential, particles are most efficiently produced in the broad resonance regimes where the amplitude of inflaton ϕ is initially large and the coupling between ϕ and a coupled scalar field χ is strong [9]. Since the expansion of the universe gradually reduces the amplitude of inflaton, this makes the χ field jump over many instability and stability bands. The χ field can grow quasiexponentially, overcoming the diluting effect by the cosmic expansion. Finally the field reaches so called narrow resonance regimes, after which particle creation terminates. In the case that the χ field is initially located in narrow resonance regimes, it is known that particle creation is inefficient because the field soon deviates from instability bands by the cosmic expansion. As for the excitation of KK modes in the models of compactifications by S^d and $S^{d_1} \times S^{d_2}$, since the σ field behaves as a massive scalar field around the minimum of its potential, we can apply analytic approaches based on the Mathieu equation [10] which have been much investigated in the context of preheating at the linear stage of the system [9,11]. However, since the investigation in Ref. [5] about the enhancement of KK modes is limited in narrow resonance regimes in which parametric resonance is weak, we will extend this work in the more general range of parameters. It is of interest whether catastrophic particle production will occur or not in the case where the value of σ is initially large and the quantum number of the angular momentum l of KK modes is not small. If KK modes are strongly enhanced by parametric resonance, this would result in some important cosmological implications. They will become good candidates for dark matter in the case that these excited particles are suitably diluted by entropy productions.

This paper is organized as follows. In the next section, we perform compactifications by a d -dimensional sphere S^d and by a direct product $S^{d_1} \times S^{d_2}$. The Casimir energy is introduced to stabilize the scale of the internal space. In Sec. III, we study the background evolution of the scale factor and the σ field, and examine the structure of resonance

for KK modes. We will investigate whether the enhancement of KK modes efficiently occurs or not both by analytic approaches and numerical integrations. We present our conclusion and discussion in the final section.

II. THE MODEL

We consider a model in $D = d + 4$ dimensions with a cosmological constant $\bar{\Lambda}$ and a single scalar field $\bar{\phi}$,

$$S = \int d^D x \sqrt{-\bar{g}} \left[\frac{1}{2\bar{\kappa}^2} \bar{R} - 2\bar{\Lambda} - \frac{1}{2} \bar{g}^{MN} \partial_M \bar{\phi} \partial_N \bar{\phi} \right], \quad (2.1)$$

where \bar{g}_{MN} and $\bar{\kappa}^2/8\pi \equiv \bar{G}$ are the D -dimensional metric and gravitational constant, respectively. \bar{R} is a scalar curvature with respect to \bar{g}_{MN} . The third term of the action (2.1) denotes a Klein-Gordon action in D dimensions.

We first compactify extra dimensions to a d -dimensional sphere S^d . Then the metric \bar{g}_{MN} can be expressed as

$$ds_D^2 = \bar{g}_{MN} dx^M dx^N = \hat{g}_{\mu\nu} dx^\mu dx^\nu + b^2 ds_d^2, \quad (2.2)$$

where $\hat{g}_{\mu\nu}$ is a four-dimensional metric, b is a scale of the d -dimensional sphere whose present value is b_0 , and ds_d^2 is a line element of the d -unit sphere. We expand the scalar field $\bar{\phi}$ on the sphere as

$$\bar{\phi} = b_0^{-d/2} \sum_{l,m} \phi_{lm} Y_{lm}^{(d)}, \quad (2.3)$$

where $Y_{lm}^{(d)}$ is the spherical harmonics on the d -sphere with positive integer l and a set of $d-1$ numbers m . Then we obtain the reduced action in four dimensions,

$$S = \int d^4 x \sqrt{-\hat{g}} \left(\frac{b}{b_0} \right)^d \left[\frac{1}{2\kappa^2} \left\{ \hat{R} + d(d-1) \frac{\partial_\mu b \partial_\nu b}{b^2} \hat{g}^{\mu\nu} + \frac{d(d-1)}{b^2} \right\} - 2V_d^0 \bar{\Lambda} \right. \\ \left. - \sum_{l,m} \left\{ \frac{1}{2} \hat{g}^{\mu\nu} \partial_\mu \phi_{lm} \partial_\nu \phi_{lm} + \frac{l(l+d-1)}{2b^2} \phi_{lm}^2 \right\} \right], \quad (2.4)$$

where $\kappa^2/8\pi = \bar{\kappa}^2/(8\pi V_d^0)$ is Newton's gravitational constant with V_d^0 the present volume of the internal space. \hat{R} is a scalar curvature related with $\hat{g}_{\mu\nu}$. Since this does not take the ordinary form of the Einstein-Hilbert action because of a time dependent factor $(b/b_0)^d$, we perform a conformal transformation as follows

$$\hat{g}_{\mu\nu} = \exp \left(-d \frac{\sigma}{\sigma_0} \right) g_{\mu\nu}, \quad (2.5)$$

where σ is the so-called dilaton field which is defined by

$$\sigma = \sigma_0 \ln \left(\frac{b}{b_0} \right), \quad (2.6)$$

$$\sigma_0 = \left[\frac{d(d+2)}{2\kappa^2} \right]^{1/2}. \quad (2.7)$$

Then the four-dimensional action in the Einstein frame can be described as

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2\kappa^2} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma - U_1(\sigma) - \frac{1}{2} \sum_{l,m} (g^{\mu\nu} \partial_\mu \phi_{lm} \partial_\nu \phi_{lm} + M_l^2(\sigma) \phi_{lm}^2) \right], \quad (2.8)$$

where R is a scalar curvature with respect to $g_{\mu\nu}$. $U_1(\sigma)$ is a potential of the σ field which we will give a specific form below, and the mass $M_l(\sigma)$ of the ϕ_{lm} field is expressed as

$$M_l^2(\sigma) = \frac{l(l+d-1)}{b_0^2} e^{-(d+2)\sigma/\sigma_0}. \quad (2.9)$$

As was mentioned in Ref. [5], the potential $U_1(\sigma)$ does not have a local minimum if we do not introduce the Casimir effect as a one-loop quantum correction. Taking into account this effect, $U_1(\sigma)$ can be written in the form

$$U_1(\sigma) = \alpha \left[\frac{2}{d+2} e^{-2(d+2)\sigma/\sigma_0} + e^{-d\sigma/\sigma_0} - \frac{d+4}{d+2} e^{-(d+2)\sigma/\sigma_0} \right], \quad (2.10)$$

with

$$\alpha = \frac{(d-1)\sigma_0^2}{(d+4)b_0^2}. \quad (2.11)$$

The first, second, and third terms in (2.10) appear due to the Casimir energy, the cosmological constant, and the curvature of the internal space, respectively. The potential $U_1(\sigma)$ acquires a local minimum at $\sigma = 0$ for the case of $d \geq 2$ because of the presence of the first term in (2.10). We depict $U_1(\sigma)$ in Fig. 1 for the case of $d = 2$. It has a local maximum at $\sigma_*(> 0)$, which depends on the extra dimension d . σ_*/σ_0 decreases with the increase of d . For example, $\sigma_*/\sigma_0 = 0.50$ for $d = 2$, and $\sigma_*/\sigma_0 = 0.22$ for $d = 6$. Since σ_0 becomes larger with the increase of d [See Eq. (2.7)], σ_* itself does not necessarily decrease with the increase of d . For example, $\sigma_* = 0.20 m_{\text{pl}}$ for $d = 2$, and $\sigma_* = 0.21 m_{\text{pl}}$ for $d = 6$. In order to result in the final state $\sigma = 0$ which corresponds to the present value $b = b_0$, initial values of σ are required to be $0 < \sigma_i < \sigma_*$ (we assume $\sigma_i > 0$), where the subscript i denotes the initial value. Then σ evolves toward the minimum of its potential, and begins to oscillate around $\sigma = 0$. In this oscillating stage, we are concerned with whether KK modes are effectively enhanced or not by parametric resonance.

In the case that the compactification is done on a direct product $S^{d_1} \times S^{d_2}$ of the d_1 -dimensional sphere S^{d_1} and d_2 -dimensional sphere S^{d_2} with $d_1 + d_2 = D - 4$, the final four-dimensional action and the potential $U_1(\sigma)$ take different forms from the S^d case. The procedure is the same as in the S^d case. First, we consider the D -dimensional metric

$$ds_D^2 = \bar{g}_{MN} dx^M dx^N = \hat{g}_{\mu\nu} dx^\mu dx^\nu + b_1^2 ds_{d_1}^2 + b_2^2 ds_{d_2}^2, \quad (2.12)$$

where b_1 and b_2 are scales of the sphere S^{d_1} and S^{d_2} , $ds_{d_1}^2$ and $ds_{d_2}^2$ are line elements of the d_1 -unit sphere and d_2 -unit sphere, respectively.

Expanding the $\bar{\phi}$ field in the D -dimensional action (2.1) as

$$\bar{\phi} = b_{10}^{-d_1/2} b_{20}^{-d_2/2} \sum_{l_1, l_2, m_1, m_2} \phi_{l_1 l_2 m_1 m_2} Y_{l_1 m_1}^{(d_1)} Y_{l_2 m_2}^{(d_2)}, \quad (2.13)$$

where b_{10} and b_{20} are present values of b_1 and b_2 , $Y_{l_1 m_1}^{(d_1)}$ and $Y_{l_2 m_2}^{(d_2)}$ are the spherical harmonics on the d_1 -sphere and d_2 -sphere, respectively, we obtain the four-dimensional action similar to the action (2.4) in the S^d case (see Ref. [5] for details). Next, we perform a conformal transformation

$$\hat{g}_{\mu\nu} = \exp \left[- \left(\frac{d_1 \sigma_1}{\sigma_{10}} + \frac{d_2 \sigma_2}{\sigma_{20}} \right) \right] g_{\mu\nu}, \quad (2.14)$$

where $\sigma_i (i = 1, 2)$ and σ_{i0} are defined by

$$\sigma_i = \sigma_{i0} \ln \left(\frac{b_i}{b_{i0}} \right), \quad (2.15)$$

$$\sigma_{i0} = \left[\frac{d_i(d_i + 2)}{2\kappa^2} \right]^{1/2}. \quad (2.16)$$

Taking into account the Casimir effect as in the S^d case and imposing the condition

$$\frac{\sigma_1}{\sigma_{10}} = \frac{\sigma_2}{\sigma_{20}} \equiv \frac{\sigma}{\sigma_0}, \quad (2.17)$$

with

$$\sigma_0 = \sqrt{\frac{(d_1 + d_2)(d_1 + d_2 + 2)}{2\kappa^2}}, \quad (2.18)$$

the final expression of the four-dimensional action yields

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2\kappa^2} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma - U_1(\sigma) - \frac{1}{2} \sum_{l_1, l_2, m_1, m_2} (g^{\mu\nu} \partial_\mu \phi_{l_1 l_2 m_1 m_2} \partial_\nu \phi_{l_1 l_2 m_1 m_2} + M_{l_1 l_2}^2(\sigma) \phi_{l_1 l_2 m_1 m_2}^2) \right], \quad (2.19)$$

where the potential of the σ field is

$$U_1(\sigma) = \alpha \left[\frac{2}{d_1 + d_2 + 2} e^{-2(d_1 + d_2 + 2)\sigma/\sigma_0} + e^{-(d_1 + d_2)\sigma/\sigma_0} - \frac{d_1 + d_2 + 4}{d_1 + d_2 + 2} e^{-(d_1 + d_2 + 2)\sigma/\sigma_0} \right], \quad (2.20)$$

with

$$\alpha = \frac{d_1 + d_2 + 2}{2(d_1 + d_2 + 4)\kappa^2} \left[\frac{d_1(d_1 - 1)}{b_{10}^2} + \frac{d_2(d_2 - 1)}{b_{20}^2} \right]. \quad (2.21)$$

The mass of $M_{l_1 l_2}(\sigma)$ of the $\phi_{l_1 l_2 m_1 m_2}$ field is given by

$$M_{l_1 l_2}^2(\sigma) = \left[\frac{l_1(l_1 + d_1 - 1)}{b_{10}^2} + \frac{l_2(l_2 + d_2 - 1)}{b_{20}^2} \right] e^{-(d_1 + d_2 + 2)\sigma/\sigma_0}. \quad (2.22)$$

The terms in the square bracket of Eq. (2.20) correspond to changing d in that of Eq. (2.10) to $d_1 + d_2$. Hence the shape of the potential $U_1(\sigma)$ in the $S^{d_1} \times S^{d_2}$ case is the same as in the S^d case. $U_1(\sigma)$ has a local minimum at $\sigma = 0$ when either of d_1 and d_2 is greater than 1. In the next section, the dynamics of σ and ϕ_{lm} fields are studied in two classes of models of (2.8) and (2.19).

III. THE DYNAMICS OF THE SYSTEM

In this section, we investigate the excitation of KK modes due to the oscillating scalar field σ by parametric resonance. We assume that the four-dimensional spacetime and the σ field are spatially homogeneous, and adopt the flat Friedmann-Robertson-Walker metric as the background spacetime

$$ds_4^2 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a^2(t) d\mathbf{x}^2, \quad (3.1)$$

where t is the cosmic time and $a(t)$ is the scale factor. The background equations for a and σ can be expressed by the action (2.8) or (2.19) as

$$\left(\frac{\dot{a}}{a} \right)^2 = \frac{\kappa^2}{3} \left[\frac{1}{2} \dot{\sigma}^2 + U_1(\sigma) \right], \quad (3.2)$$

$$\ddot{\sigma} + 3 \frac{\dot{a}}{a} \dot{\sigma} + U_1'(\sigma) = 0, \quad (3.3)$$

where we have neglected the contributions from KK modes. However, in the case that KK modes are enhanced significantly, this affects the evolution of background equations. We briefly comment on this back reaction issue at the end of this section.

As we have already mentioned, the initial value of σ is required to be smaller than σ_* in order not to lead to an expansion of the internal space. Let us consider the case where σ is initially located close to σ_* with $\sigma_i < \sigma_*$. In this case, defining a dimensionless parameter

$$\epsilon = \begin{cases} (d+2) \frac{\sigma}{\sigma_0} & (S^d \text{ case}), \\ (d_1 + d_2 + 2) \frac{\sigma}{\sigma_0} & (S^{d_1} \times S^{d_2} \text{ case}), \end{cases} \quad (3.4)$$

it usually exceeds of the order of unity at the beginning with $d \geq 2$ or $d_1 + d_2 \geq 2$. For example, in the S^d case, $\epsilon(t_i) = 2.0$ when $d = 2$, and $\epsilon(t_i) = 1.8$ when $d = 6$. This is the situation which was not considered in Ref. [5]. After

a short stage of rolling down, the σ field begins to oscillate around the minimum of its potential at $\sigma = 0$. After that, the σ field behaves as a massive scalar field whose mass is given by

$$m_\sigma \equiv \sqrt{U_1''(0)} = \begin{cases} \frac{\sqrt{2(d-1)}}{b_0} & (S^d \text{ case}), \\ \sqrt{\frac{2}{d_1+d_2} \left[\frac{d_1(d_1-1)}{b_{10}^2} + \frac{d_2(d_2-1)}{b_{20}^2} \right]} & (S^{d_1} \times S^{d_2} \text{ case}). \end{cases} \quad (3.5)$$

The evolution of this field is the same as a massive inflaton field ϕ with a chaotic potential $V(\phi) = m_\phi^2 \phi^2/2$ in the reheating phase. Making use of the time-averaged relation

$$\frac{1}{2} \langle \dot{\sigma}^2 \rangle = \langle U_1(\sigma) \rangle, \quad (3.6)$$

we easily find from Eqs. (3.2) and (3.3) that the evolution of a and σ can be approximately written as

$$a = \left(\frac{t}{t_{\text{co}}} \right)^{2/3}, \quad (3.7)$$

$$\sigma(t)/\sigma_0 = \tilde{\sigma}(t) \cos m_\sigma t, \quad (3.8)$$

where t_{co} denotes the time when the coherent oscillation of σ starts, and $\tilde{\sigma}(t)$ is the dimensionless amplitude of the σ field which decreases as $\tilde{\sigma}(t) \sim 1/t$ by the cosmic expansion. Eq. (3.7) shows that the universe evolves as matter dominant when the σ field oscillates coherently. Hence the energy density of σ decreases as $\rho_\sigma \sim 1/t^3$.

Let us study the excitation of KK modes during the oscillating stage of the σ field. For each compactification of the S^d and $S^{d_1} \times S^{d_2}$ cases, we expand ϕ_{lm} (S^d case) and $\phi_{l_1 l_2 m_1 m_2}$ ($S^{d_1} \times S^{d_2}$ case) fields by the Fourier component as

$$\phi_{lm} = \frac{1}{(2\pi)^{3/2}} \int \left(a_k \phi_{lm}^{\mathbf{k}}(t) e^{-i\mathbf{k} \cdot \mathbf{x}} + a_k^\dagger \phi_{lm}^{\mathbf{k}*}(t) e^{i\mathbf{k} \cdot \mathbf{x}} \right) d^3 \mathbf{k}, \quad (3.9)$$

$$\phi_{l_1 l_2 m_1 m_2} = \frac{1}{(2\pi)^{3/2}} \int \left(a_k \phi_{l_1 l_2 m_1 m_2}^{\mathbf{k}}(t) e^{-i\mathbf{k} \cdot \mathbf{x}} + a_k^\dagger \phi_{l_1 l_2 m_1 m_2}^{\mathbf{k}*}(t) e^{i\mathbf{k} \cdot \mathbf{x}} \right) d^3 \mathbf{k}, \quad (3.10)$$

where a_k and a_k^\dagger are the annihilation and creation operators, respectively. Then we find that the temporary parts $\phi_{lm}^{\mathbf{k}}(t)$ and $\phi_{l_1 l_2 m_1 m_2}^{\mathbf{k}}(t)$ obey the following equations of motion:

$$\ddot{\phi}_{lm}^{\mathbf{k}} + 3 \frac{\dot{a}}{a} \dot{\phi}_{lm}^{\mathbf{k}} + \left[\frac{k^2}{a^2} + \frac{l(l+d-1)}{b_0^2} e^{-(d+2)\sigma/\sigma_0} \right] \phi_{lm}^{\mathbf{k}} = 0 \quad (S^d \text{ case}), \quad (3.11)$$

$$\begin{aligned} & \ddot{\phi}_{l_1 l_2 m_1 m_2}^{\mathbf{k}} + 3 \frac{\dot{a}}{a} \dot{\phi}_{l_1 l_2 m_1 m_2}^{\mathbf{k}} + \\ & \left[\frac{k^2}{a^2} + \left\{ \frac{l_1(l_1+d_1-1)}{b_{10}^2} + \frac{l_2(l_2+d_2-1)}{b_{20}^2} \right\} e^{-(d_1+d_2+2)\sigma/\sigma_0} \right] \phi_{l_1 l_2 m_1 m_2}^{\mathbf{k}} = 0 \quad (S^{d_1} \times S^{d_2} \text{ case}). \end{aligned} \quad (3.12)$$

Hereafter, we express both fields $\phi_{lm}^{\mathbf{k}}$ and $\phi_{l_1 l_2 m_1 m_2}^{\mathbf{k}}$ as ϕ_k except the case that distinctions of both fields are required.

Defining a new scalar field $\varphi_k = a^{3/2} \phi_k$, Eqs. (3.11) and (3.12) are written as

$$\ddot{\varphi}_k + \omega_k^2 \varphi_k = 0, \quad (3.13)$$

where for the S^d case,

$$\omega_k^2 \equiv \frac{k^2}{a^2} + \frac{l(l+d-1)}{b_0^2} e^{-(d+2)\sigma/\sigma_0} - \frac{3}{4} \left(\frac{2\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right), \quad (3.14)$$

and for the $S^{d_1} \times S^{d_2}$ case,

$$\omega_k^2 \equiv \frac{k^2}{a^2} + \left\{ \frac{l_1(l_1+d_1-1)}{b_{10}^2} + \frac{l_2(l_2+d_2-1)}{b_{20}^2} \right\} e^{-(d_1+d_2+2)\sigma/\sigma_0} - \frac{3}{4} \left(\frac{2\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right). \quad (3.15)$$

As for the initial conditions of the φ_k field, we choose the state that corresponds to the conformal vacuum as

$$\varphi_k(t_i) = \frac{1}{\sqrt{2\omega_k(t_i)}}, \quad \dot{\varphi}_k(t_i) = -i\omega_k(t_i)\varphi_k(t_i). \quad (3.16)$$

The second terms in Eqs. (3.14) and (3.15) are resonance terms which would lead to the amplification of KK modes.

When ϵ defined by Eq. (3.4) is greater than of the order of unity, resonance terms are suppressed and do not play relevant roles for the enhancement of the φ_k field. However, ϵ decreases under unity and σ begins to oscillate coherently, we can expect the parametric resonance phenomenon. In this case, Eq. (3.13) is reduced to the Mathieu equation,

$$\frac{d^2}{dz^2} \varphi_k + (A_k - 2q \cos 2z) \varphi_k = 0, \quad (3.17)$$

where for the S^d case,

$$A_k = \frac{2l(l+d-1)}{d-1} + 4 \frac{(k/m_\sigma)^2}{a^2}, \quad (3.18)$$

$$q = \frac{l(l+d-1)}{d-1} (d+2) \tilde{\sigma}(t), \quad (3.19)$$

and for the $S^{d_1} \times S^{d_2}$ case,

$$A_k = \frac{2(d_1+d_2)\{l_1(l_1+d_1-1) + l_2(l_2+d_2-1)(b_{10}/b_{20})^2\}}{d_1(d_1-1) + d_2(d_2-1)(b_{10}/b_{20})^2} + 4 \frac{(k/m_\sigma)^2}{a^2}, \quad (3.20)$$

$$q = \frac{(d_1+d_2)\{l_1(l_1+d_1-1) + l_2(l_2+d_2-1)(b_{10}/b_{20})^2\}}{d_1(d_1-1) + d_2(d_2-1)(b_{10}/b_{20})^2} (d_1+d_2+2) \tilde{\sigma}(t). \quad (3.21)$$

z is defined as $z = mt$. We have neglected the contribution of the third term in Eq. (3.14) and (3.15) since this term is not important in the oscillating stage of the σ field [9].

We can analyze the excitation of KK modes by making use of the stability and instability chart of the Mathieu equation [10]. Parametric resonance occurs when KK modes stay in the instability regions sketched in Fig. 2. These regions are typically described by narrow resonance regimes of $q \lesssim 1$ and broad resonance regimes of $q \gg 1$. As is found in Fig. 2, the instability bands are broader for larger values of q as long as A_k is not so large. Generally, particle creation in narrow resonance regimes is not so efficient as in the case of broad resonance regimes [9]. Since the past work about the excitation of KK modes [5] is limited in narrow resonance regimes $q \lesssim 1$, we extend this work in the more general case with $q \gg 1$. However, note that the results obtained in the context of preheating with a quadratic potential [9] are not directly applied to the present case, because the relation of A_k and q is different.

First, let us first examine the S^d case. As is found in Eq. (3.18), A_k approaches the constant value

$$B \equiv \frac{2l(l+d-1)}{(d-1)}, \quad (3.22)$$

with the passage of time for fixed values of d and l . Since B is greater than unity for the positive integer d and l with $d \geq 2$, parametric resonance does not occur in the first resonance band around $A_k = 1$ as was pointed out in Ref. [5]. For example, when $d = 2$ and $l = 1$, $B = 4$. The initial value of q ($= q_{\text{co}}$) at which the analysis based on the Mathieu equation becomes valid depends on the value of σ when the σ field begins to oscillate coherently. Numerical calculations show that this corresponds to $\epsilon(t_{\text{co}}) \approx 0.4$, after which the evolution of the σ field can be described as Eq. (3.8). This means that the $(d+2)\tilde{\sigma}(t)$ term in Eq. (3.19) and the $(d_1+d_2+2)\tilde{\sigma}(t)$ term in Eq. (3.21) are restricted to be smaller than unity. In order to realize the situation $q \gg 1$, we need to choose larger values of l , or l_1 and l_2 .

Let us investigate two cases of $q_{\text{co}} \lesssim 1$ and $q_{\text{co}} \gg 1$. First, we consider the case of $q_{\text{co}} \lesssim 1$ with the extra dimension $d = 2$. When $l = 1$, since $A_k = 4 + 4(k/m_\sigma a)^2$ and $q_{\text{co}} = 8\tilde{\sigma}(t_{\text{co}}) \approx 0.8$ by Eqs. (3.18) and (3.19), the φ_k field is initially located in the second instability band around $A_k = 4$ for low momentum modes. However, the value of q becomes smaller with the decrease of the amplitude $\tilde{\sigma}(t)$ as $\tilde{\sigma}(t) \sim 1/t$ by the cosmic expansion. In Fig. 3, we depict the evolution of the σ field in this case. We choose $\sigma(t_i)/\sigma_0 = 0.4$ as the initial value of the σ field. After the first stage of rolling down, σ begins to oscillate with the initial amplitude $\tilde{\sigma}(t_{\text{co}}) \approx 0.1$. The coherent oscillation starts by the same amplitude as long as the initial value of σ is located in the region of $0.1 \lesssim \sigma(t_i)/\sigma_0 \lesssim \sigma_*/\sigma_0 \approx 0.5$. After several oscillations, the amplitude drops down $\tilde{\sigma}(t) \lesssim 0.02$, which corresponds to $q \lesssim 0.1$. Although the φ_k field stays in the instability band since A_k approaches $A_k \rightarrow 4$, the growth rate μ_k of the φ_k field becomes smaller with the decrease of q . In the j -th resonance band with $A_k = j^2$, $q \ll 2j^{3/2}$, and $j \geq 2$, μ_k is expressed as [10]

$$\mu_k = \frac{\sin 2\delta}{2j[2^{j-1}(j-1)]^2} q^j, \quad (3.23)$$

where δ changes in the interval $[0, \pi/2]$. In the second instability band with $q \leq q_{\text{co}} \approx 0.8$, $\mu_k \propto q^2$. The growth rate μ_k is initially of the order 0.01, and after that μ_k rapidly decreases. The growth rate of KK modes $\phi_k = a^{-3/2}\varphi_k$ is approximately expressed as

$$\frac{d\phi_k}{dz} \approx a^{-3/2} \left(\mu_k - \frac{3}{2} \frac{H}{m} \right) \varphi_k, \quad (3.24)$$

where we have used the relation $\varphi_k \sim e^{\mu_k z}$. Since the Hubble expansion rate at $t = t_{\text{co}}$ is $H/m \approx 0.28$, which exceeds the growth rate of φ_k , ϕ_k decreases at the beginning. Moreover, since μ_k and H decrease as $\mu_k \propto q^2 \propto 1/t^2$ and $H \propto 1/t$, the increase of ϕ_k can not be expected in the second instability band of the narrow resonance regime $q \lesssim 1$. We have numerically confirmed that ϕ_k does not increase in the case of $d = 2$ and $l = 1$ (see Fig. 4). If there is a set of d and l which satisfies $B = j^2$ with positive integer $j \geq 3$, the narrow instability bands exist around the regions of $A_k = j^2$. However, parametric resonance is much more inefficient than in the second resonance case, since the growth rate μ_k decreases with the increase of j for $q \lesssim 1$ as is found by Eq. (3.23). Even if d and l are changed, the first resonance does not occur because B defined in Eq. (3.22) is always greater than unity. In the narrow resonance of $q \lesssim 1$ with the S^d compactification, we can conclude that KK modes are not enhanced sufficiently in any values of the extra dimension d and the positive integer l .

Consider the case of $q \gg 1$ with the S^d compactification. As we have already mentioned, larger values of l are required to lead to the condition $q \gg 1$. In order to judge whether parametric resonance is efficient or not, we should know the relation between A_k and q . In the S^d case, we obtain the following relation from Eqs. (3.18) and (3.19),

$$A_k = \frac{2}{(d+2)\tilde{\sigma}(t)} q + 4 \frac{(k/m_\sigma)^2}{a^2}. \quad (3.25)$$

The efficiency of resonance strongly depends on the value of $(d+2)\tilde{\sigma}(t)$ in Eq. (3.25). We have numerically confirmed that this value is approximately $(d+2)\tilde{\sigma}(t_{\text{co}}) \approx 0.4$ when the σ field begins to oscillate coherently for any values of d . This corresponds the relation of $A_k \approx 5q_{\text{co}} + 4(k/m_\sigma a)^2$ at the beginning of the oscillating stage. Although there are many instability bands in the regions of $q \gg 1$ which are generally called broad resonance bands, these bands become rare in the regions of $A_k \gtrsim 5q$ (see Fig. 2). Moreover, the tangent of the line $A_k = 2q/\{(d+2)\tilde{\sigma}(t)\}$ gets larger with the passage of time because of the decrease of the amplitude $\tilde{\sigma}(t)$ as $\tilde{\sigma}(t) \sim 1/t$, which means that parametric resonance becomes inefficient. Even in the case that the φ_k field is initially in an instability band, the

expansion of the universe makes the field deviate from the instability band. In the model of a massive inflaton with a coupled scalar field χ in preheating after inflation, the χ field moves on the path of $A_k = 2q + k^2/(m_\phi^2 a^2)$ with the decrease of q as $q \sim 1/t^2$. In this case, there are many instability bands as well as stability bands in the regions of $A_k \geq 2q$, the χ field *stochastically* increases when passing through instability bands. Since the tangent of the A_k - q path does not change, χ particles can be efficiently produced overcoming the diluting effect by the cosmic expansion [9]. In the present model, the excitation of KK modes is very weak because the regions of $A_k \geq 2q/\{(d+2)\tilde{\sigma}(t)\}$ are mostly occupied by stability bands. Although the φ_k field passes through instability bands, these are very few in the regions of $A_k \geq 2q/\{(d+2)\tilde{\sigma}(t)\}$, and parametric resonance is not efficient. Even the low momentum modes close to the line $A_k = 2q/\{(d+2)\tilde{\sigma}(t)\}$ are not enhanced sufficiently. As time passes, both A_k and q decrease by the cosmic expansion, and finally approaches $A_k = B = 2l(l+d-1)/(d-1)$ and $q = 0$. In Fig. 5, we show the evolution of the KK mode ϕ_k with $d = 2$, $l = 100$, and $k = 0$. In this case, $q_{\text{co}} \approx 8000$. We find that the KK mode does not grow as in the case of $q \lesssim 1$. Namely, the growth rate μ_k of the φ_k field does not surpass the Hubble expansion rate even in the case of $q \gg 1$. As we have already mentioned, the initial relation of A_k and q hardly depends on the values of $\sigma(t_{\text{co}})$ and d , which means that parametric resonance is also inefficient even if we choose larger values of $\sigma(t_i)$ and d .

Next, let us consider the compactification by $S^{d_1} \times S^{d_2}$. In this case, since the value

$$B \equiv \frac{2(d_1 + d_2)\{l_1(l_1 + d_1 - 1) + l_2(l_2 + d_2 - 1)(b_{10}/b_{20})^2\}}{d_1(d_1 - 1) + d_2(d_2 - 1)(b_{10}/b_{20})^2} \quad (3.26)$$

is greater than unity for positive integers l_1, l_2 and d_1, d_2 with $d_1 \geq d_2$ and $d_1 + d_2 = D - 4$, the first resonance does not occur as in the case of S^d [12]. When B is written by a positive integer $j (\geq 2)$ as $B = j^2$ and the value of q_{co} is smaller than unity, the φ_k field is located in an instability band at the initial stage. However, since q decreases as $q \sim 1/t$ by the cosmic expansion and the growth rate of KK modes is very small for $j \geq 2$, the excitation of KK modes is inefficient in narrow resonance regimes $q \lesssim 1$. In the case of $q \gg 1$, since the relation of A_k and q is written by Eqs. (3.20) and (3.21) as

$$A_k = \frac{2}{(d_1 + d_2 + 2)\tilde{\sigma}(t)}q + 4\frac{(k/m_\sigma)^2}{a^2}, \quad (3.27)$$

the discussions in the S^d case can be applied by changing $d_1 + d_2$ to d . We have numerically confirmed that the σ field begins to oscillate coherently at $(d_1 + d_2 + 2)\tilde{\sigma}(t_{\text{co}}) \approx 0.4$ for any values of $\tilde{\sigma}(t_i)$, and d_1, d_2 . Moreover, since $\tilde{\sigma}(t)$ decreases as $\tilde{\sigma}(t) \sim 1/t$, parametric resonance is also ineffective as in the S^d case.

We conclude that catastrophic particle creation does not occur in two classes of models by S^d and $S^{d_1} \times S^{d_2}$

compactifications in any values of parameters. As a result, the back reaction effect by KK modes is not important in these models.

IV. CONCLUDING REMARKS AND DISCUSSIONS

In this paper, we have studied the excitement of Kaluza-Klein (KK) modes in a higher D -dimensional generalized Kaluza-Klein theory. We have considered two classes of models where the extra dimensions are compactified on the sphere S^d with $d = D - 4$ and the direct product $S^{d_1} \times S^{d_2}$ with $d_1 + d_2 = D - 4$. Such compactifications give rise to a term which originates from the curvature of the internal space in the potential $U_1(\sigma)$ of a dilaton field σ . Since this potential does not have a local minimum to stabilize the scale of the internal space, we introduce the Casimir effect due to a one-loop quantum correction. Then the σ field oscillates as a massive scalar field around the local minimum at $\sigma = 0$, which corresponds to the present value of the scale of compactifications.

Since the KK field acquires a mass which can be expressed by the σ field by compactifications on S^d and $S^{d_1} \times S^{d_2}$, we can expect that KK modes will be enhanced by parametric resonance. The past work on this issue [5] is restricted in the case of narrow resonance regimes where the resonance parameter q is smaller than unity. However, in the similar situation of preheating after inflation with a quadratic potential, it is well known that resonance with $q \gg 1$ is much more efficient than in the case of $q \lesssim 1$. Hence we extend past work on the excitation of KK modes to the case of $q \gg 1$ by making use of the stability and instability chart of the Mathieu equation.

In the case of $q \lesssim 1$, the first resonance does not occur for both compactifications by S^d and $S^{d_1} \times S^{d_2}$. Although there are some situations where the KK field is located in the second, third, \dots instability bands at the beginning of the coherent oscillation of the σ field, parametric resonance soon becomes ineffective with the decrease of q due to the expansion of the universe. In this case, since the creation rate of KK modes can not surpass the Hubble expansion rate, the enhancement of KK modes is inefficient.

Even in the case of $q \gg 1$, we have found that the growth of KK modes does not take place both by analytic approaches and numerical integrations. The σ field begins to oscillate coherently when the amplitude of the σ field drops down to $\epsilon \approx 0.4$, where ϵ is defined by (3.4). Since the relation of resonance parameters A_k and q are expressed as $A_k = 2q/\{(d+2)\tilde{\sigma}(t)\} + 4(k/m_\sigma a)^2$ for the S^d case, and $A_k = 2q/\{(d_1+d_2+2)\tilde{\sigma}(t)\} + 4(k/m_\sigma a)^2$ for the $S^{d_1} \times S^{d_2}$ case, the KK field exists in the region of $A_k \gtrsim 5q$ where instability bands are few at the beginning. The amplitude of σ decreases with the passage of time, and the KK field evolves in the regions where instability bands are further

few. As a result, KK modes are not relevantly enhanced even for the case of $q \gg 1$. We have numerically confirmed this fact, and found that the excitation of KK modes is inefficient in any parameters in two classes of models of compactifications.

Since we find that KK modes are not overproduced by parametric resonance by compactifications of S^d and $S^{d_1} \times S^{d_2}$, this kind of compactification may not be ruled out from a cosmological point of view, because the energy density of KK modes will not overclose the universe in the radiation dominant era. However, at the stage of preheating after inflation, scalar fields coupled to an inflaton field can be strongly enhanced by parametric resonance [9]. If the KK field is coupled to inflaton, the enhancement of KK modes would also occur in the preheating stage. Recently, Mazumdar and Mendes [13] considered the excitement of the dilaton field σ as well as the Brans-Dicke field during the preheating phase in generalized Einstein theories. Since they compactified the extra dimensions on torus which does not have the curvature of the internal space, the potential of dilaton $U_1(\sigma)$ does not appear in their model. Taking into account the growth of metric perturbations during preheating [14], it was found that the dilaton field σ can be effectively enhanced even when dilaton does not couple to inflaton. Although they did not consider the enhancement of KK modes, there will be a possibility that KK modes are strongly enhanced in the preheating phase even in the case where the KK field does not directly couple to inflaton by the growth of metric perturbations. In this model, back reaction effects would play an important role for the termination of resonance. Although it is technically difficult to deal with back reaction issues including second order metric perturbations in a consistent way, it is of interest how KK modes are enhanced in the preheating phase. These issues are under consideration.

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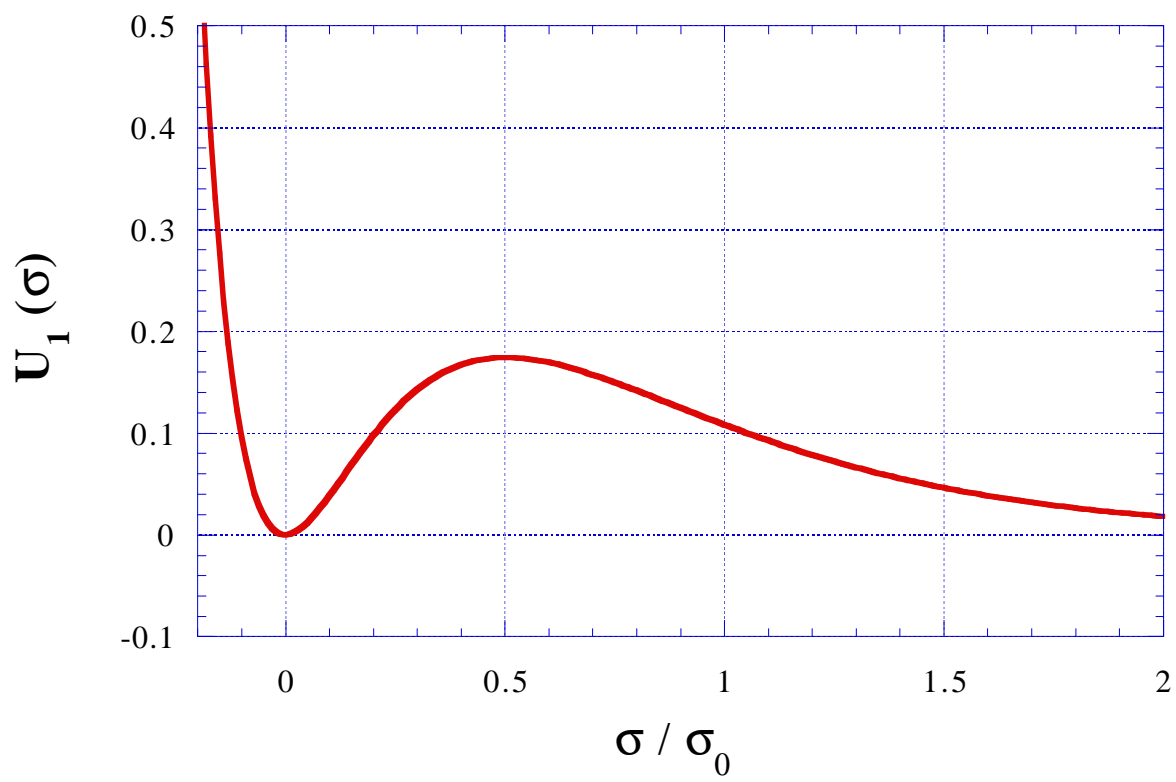
The author would like to thank Kei-ichi Maeda, Takashi Torii, Kunihiro Uzawa, and Hiroki Yajima for useful discussions. This work was supported partially by a Grant-in-Aid for Scientific Research Fund of the Ministry of Education, Science and Culture (No. 09410217), and by the Waseda University Grant for Special Research Projects. Recently, Uzawa, Morisawa, and Mukohyama [15] considered the excitement of Kaluza Klein modes including metric perturbations in the narrow resonance case, and found that quanta of these modes are not enhanced sufficiently. This result is consistent with our results obtained in this paper.

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- [1] Th. Kaluza, Sitzungsber. Preuss. Akad. Wiss. Berlin, Math. Phys. K1, 966 (1921); O. Klein, Z. Phys. 37, 895 (1926).
 - [2] M. B. Green, J. H. Schwartz, and E. Witten, *Superstring theory* (Cambridge University Press, Cambridge, England, 1987).
 - [3] E. Witten, Nucl. Phys. **B443** (1995); P. Horava and E. Witten, Nucl. Phys. **B460** (1996).
 - [4] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999); L. Randall and R. Sundrum, hep-th/9906064.
 - [5] S. Mukohyama, Phys. Rev. D **57**, 6191 (1998).
 - [6] E. W. Kolb and R. Slansky, Phys. Lett. **135B**, 378 (1984).
 - [7] The stability of internal dimensions are discussed by several authors. For example, see K. Maeda, Phys. Lett. **186B**, 33 (1987); L. Amendola, E. W. Kolb, M. Litterio, and F. Occhionero, Phys. Rev. D **42**, 1944 (1990).
 - [8] J. Traschen and R. H. Brandenberger, Phys. Rev. D **42**, 2491 (1990); Y. Shatanov, J. Traschen, and R. H. Brandenberger, Phys. Rev. D **51**, 5438 (1995).
 - [9] L. Kofman, A. Linde, and A. A. Starobinsky, Phys. Rev. Lett. **73**, 3195 (1994); L. Kofman, A. Linde, and A. A. Starobinsky, Phys. Rev. D **73**, 3258 (1997).
 - [10] N. W. Mac Lachlan, *Theory and Applications of Mathieu Functions* (Dover, New York, 1961).
 - [11] S. Tsujikawa, K. Maeda, and T. Torii, Phys. Rev. D **60**, 063515 (1999); S. Tsujikawa, K. Maeda, and T. Torii, Phys. Rev. D **60**, 123505 (1999).
 - [12] Although the case of $B \leq 1$ is considered in Ref. [5], we can easily verify that this does not hold for positive integers l_1, l_2 and d_1, d_2 .
 - [13] A. Mazumdar and L. E. Mendes, Phys. Rev. D **60**, 103513 (1999).
 - [14] A. Taruya and Y. Nambu, Phys. Lett. B **428**, 37 (1997); B. A. Bassett, D. I. Kaiser, and R. Maartens, Phys. Lett. **B455**, 84 (1999); B. A. Bassett, F. Tamburini, D. I. Kaiser, and R. Maartens, hep-ph/9901319, to appear in Nucl. Phys. B; F. Finelli and R. Brandenberger, Phys. Rev. Lett. **82** 1362 (1999); M. Parry and R. Easther, Phys. Rev. D **59** 061301 (1999).
 - [15] K. Uzawa, Y. Morisawa, and S. Mukohyama, gr-qc/9912108.

Figure Captions

- FIG. 1: The potential $U_1(\sigma)$ which is obtained by introducing the Casimir effect in the S^d compactification with $d = 2$. The potential has a minimum at $\sigma = 0$ and a local maximum at $\sigma_*/\sigma_0 = 0.50$. If we choose larger values of d , σ_*/σ_0 becomes smaller. The shape of this potential in the case of the $S^{d_1} \times S^{d_2}$ compactification is the same as in the case of the S^d compactification.
- FIG. 2: The schematic diagram of the Mathieu chart. The lined regions denote the instability bands. There exists narrow instability bands around $A_k = j^2$ and $q < 1$ with positive integer j . Although there are many instability bands in the regions of $q \gg 1$, they are few for $A_k \geq 5q$.
- FIG. 3: The evolution of σ as a function of t in the case of the S^d compactification with $d = 2$, $l = 1$ and $k = 0$. We choose the initial value of σ as $\sigma/\sigma_0 = 0.4$. After the first stage of rolling down, the σ field begins to oscillate coherently as Eq. (3.8). The dimensionless amplitude $\tilde{\sigma}(t)$ in Eq. (3.8) decreases as $\tilde{\sigma}(t) \sim 1/t$ with the initial value of $\tilde{\sigma}(t_{co}) \approx 0.1$.
- FIG. 4: The evolution of the real part of the Kaluza-Klein mode ϕ_k as a function of t in the case of the S^d compactification with $d = 2$, $l = 1$, and $k = 0$. Although the ϕ_k field is initially in the second instability band with $q \lesssim 1$, the expansion of the universe makes parametric resonance ineffective. As a result, the growth of ϕ_k can not be expected.
- FIG. 5: The evolution of the real part of the Kaluza-Klein mode ϕ_k as a function of t in the case of the S^d compactification with $d = 2$, $l = 100$, and $k = 0$. Although the initial value of q is large as $q \gg 1$, the ϕ_k field moves in the regions of $A_k \geq 2q/\{(d+2)\tilde{\sigma}(t)\}$ where instability bands are few. Hence parametric resonance is inefficient.

Fig. 1



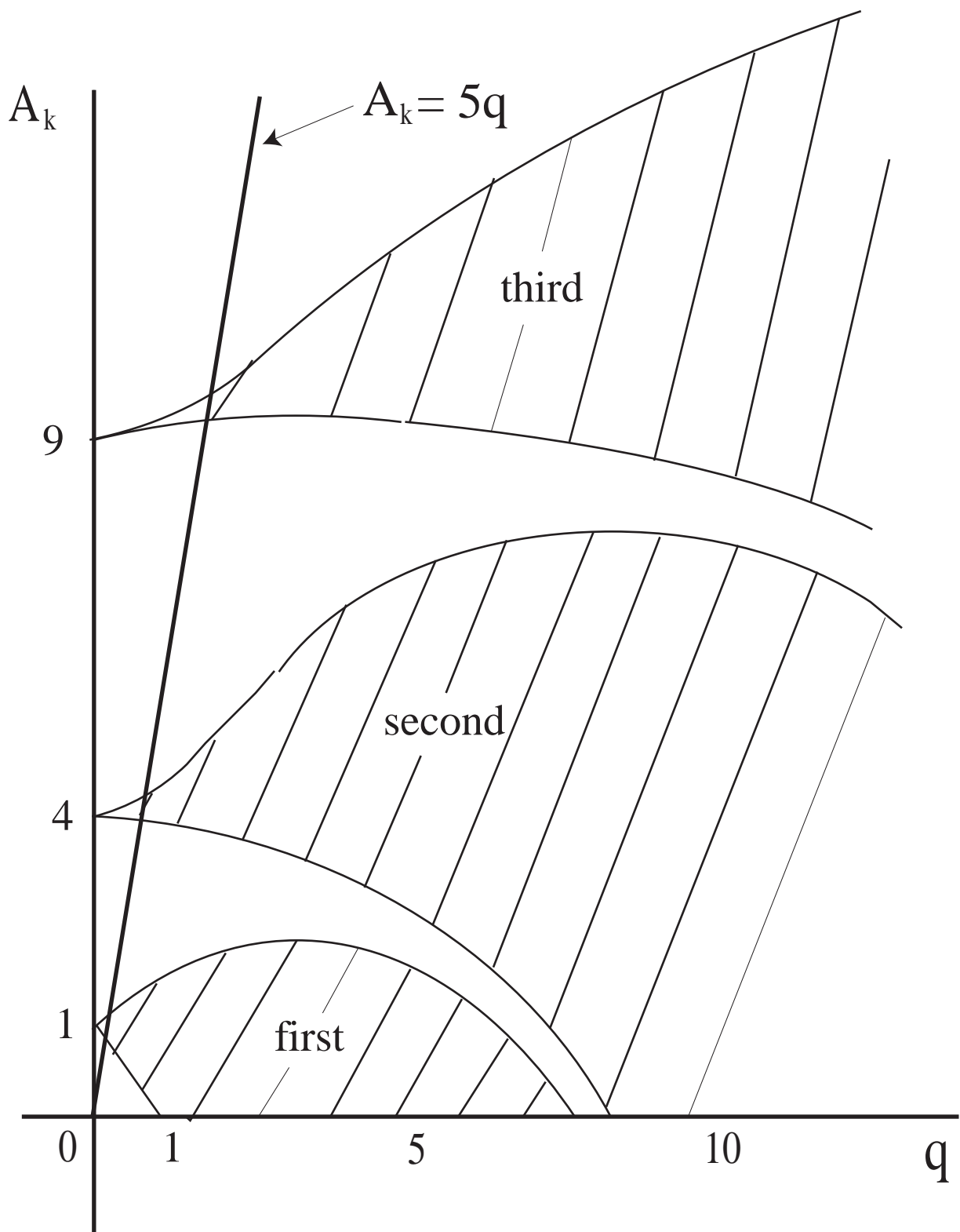


Fig. 2

Fig. 3

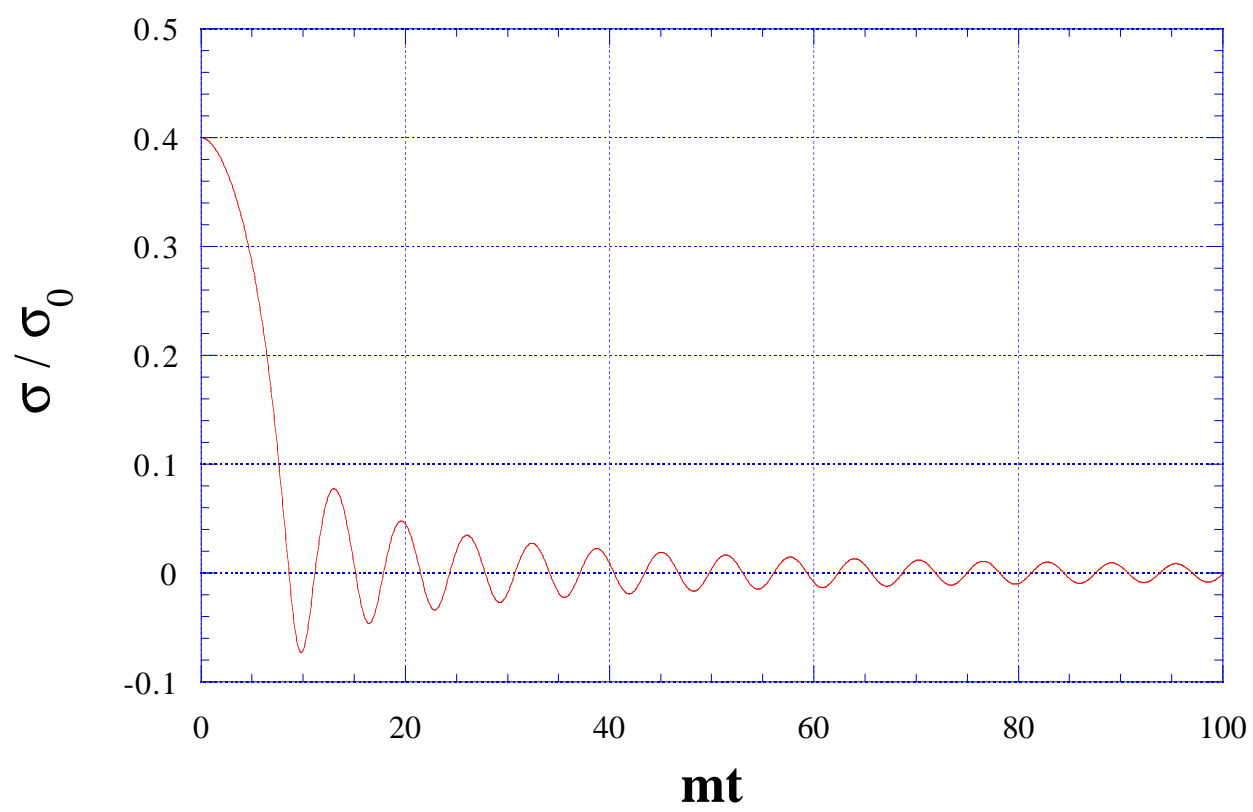


Fig. 4

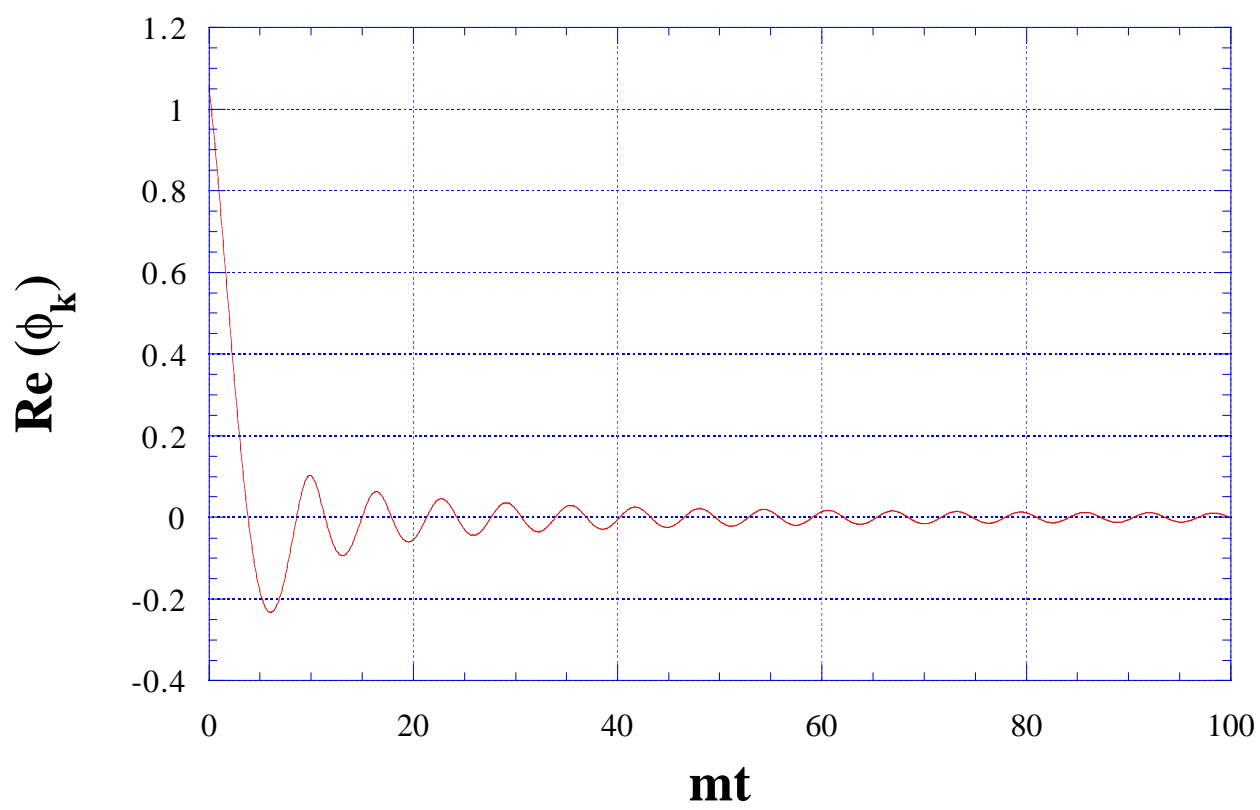


Fig. 5

